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An Alternative Evaluation of the Integral of Bessel-Gauss Beam

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Abstract
The integral solution used to obtain the Bessel-Gauss beam is revisited using the analytical method. It is seen that an alternative form of representation of the Bessel-Gauss beam is achieved other than in the literature.

Review Article

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1. Introduction

Unlike most beams, which are the solutions of paraxial wave equations, such as the Gauss beam, the Bessel beam is a solution of the Helmholtz equation (Yalçın, 2010). It is possible to reduce the Helmholtz equation to a paraxial wave equation for optical wave propagation analysis (Durnin, 1987). Bessel beam has a non-diffracting property (Yalçın, 2019). This means that when it propagates, it does not spread in the transverse plane. In addition, the Bessel beam has another essential feature, which is self-healing (Chu, 2012). This property is related to subcomponents where the Bessel beam is composed of two oppositely propagating Hankel beams (Basdemir, 2024). Bessel beam has an infinite amount of energy in the theory. However, to generate a Bessel beam experimentally, optical elements such as axicons are used (Brzobohatý et al., 2008; Williams and Pendry, 2005). This process limits its energy. Therefore, new beam types called Bessel-Gauss are revealed. Theoretically, the Bessel-Gauss beam is obtained by integrating the Huygens-Fresnel integral at $z = 0$ plane with an expression including the Bessel expression (Gori et al. 1987). They used the expression 6.615, which is given in (Gradshteyn and Ryzhik, 2007). In the evaluated Bessel-Gauss beam expression, when z is set to zero, the field distribution in the *z-axis* yields zero value. This can be connected to the deficiency of the expression 6.615 Gradshteyn and Ryzhik, 2007). In this study, an alternative result is derived using the Hankel transform and convolution property instead of this expression.

2. Theory

The Fresnel diffraction integral expression, which is related to the Bessel-Gauss, is written as

$$\int_0^\infty e^{-\alpha x} J_\nu(2\beta\sqrt{x}) J_\nu(2\gamma\sqrt{x})dx = \frac{1}{\alpha} I_\nu\left(\frac{2\beta\gamma}{\alpha}\right) exp\left(-\frac{\beta^2+\gamma^2}{\alpha}\right) \quad (1)$$

where α, β and γ are possible complex parameters for any choice and Iν is the first kind modified Bessel function given in (Gradshteyn and Ryzhik 2007). The new variables can be defined as

$$\sqrt{x} = u, \quad (2)$$

$$x = u^2, \quad (3)$$

and

$$dx = 2udu \quad (4)$$

for evaluation of the Eq. (1). Equation (1) is rewritten as

$$I = 2\int_0^\infty e^{-\alpha u^2} J_\nu(2\beta u) J_\nu(2\gamma u) udu (5)$$

taking into account Eq. (2), (3), and (4). The main idea, which is the evaluation of Eq. (1), is based on the Hankel transform and convolutional properties. Let $f(x)$ be a function defined for $x \geq 0$. The νth order Hankel transform of $f(x)$ is defined as

$$H_\nu\{f(x)\} \equiv \int_0^\infty x f(x) J_\nu(sx)dx \quad (6)$$

from a reference book(C. A. Balanis 2012). The integral I is divided into two sub-integrals. The first part is written as

$$II = \int_0^\infty J_\nu(2\beta u) e^{-\alpha u^2} udu \quad (7)$$

according to the Hankel transform. The variables have to be changed for evaluation of the Eq. (7). The new variables are defined as

$$u^2 = x, \quad (8)$$

$$u = \sqrt{x}, \quad (9)$$

and

$$2udu = dx. \quad (10)$$

which are related with the Eq. (7). The Eq. (7) is rearranged as

$$II = \frac{1}{2}\int_0^\infty J_\nu(2\beta\sqrt{x}) e^{-\alpha x} dx \quad (11)$$

with using Eq. (8), Eq. (9), and Eq. (10). The integral expression can be written

$$\int_0^\infty e^{-\alpha x} J_\nu(\beta\sqrt{x})dx = \frac{1}{\alpha} e^{-\frac{\beta^2}{4\alpha}} \quad (12)$$

from the Ref. (Gradshteyn and Ryzhik 2007) where the expression number is 6.614. When Eq. (12) is taken into account, Eq. (11) is evaluated as

$$II = \frac{1}{2\alpha} e^{-\frac{\beta^2}{\alpha}}. \quad (13)$$

The second part of the Eq. (5) is written as

$$III = \int_0^\infty J_\nu(2\beta u) J_\nu(2\gamma u) udu \quad (14)$$

according to the Hankel transform (C. A. Balanis 2012). The integral expression which is used for the evaluation of the Eq. (14) is given as

$$\int_0^\infty kJ_n(ka)J_n(kb)dk = \frac{1}{a}\delta(b-a) \quad (15)$$

from the Ref. (Gradshteyn and Ryzhik 2007), the expression number is 6.512-8. The evaluated expression of the Eq. (14) is written as

$$III = \frac{1}{2\gamma}\delta(2\beta - 2\gamma) \quad (16)$$

where $\delta(x)$ is the direct delta function. The final case of the Eq. (5) can be found as

$$I = 2(II * III) \quad (17)$$

where * is the convolutional operation (C. A. Balanis 2012). Convolutional integral is written as

$$I = 2\int_{-\infty}^{\infty} \frac{1}{2\gamma}\delta(2\tau - 2\gamma)\frac{1}{2\alpha}e^{-\frac{(\beta-\tau)^2}{\alpha}}d\tau. \quad (18)$$

The evaluation of the Eq. (18) is possible by using the variable transformation. The new variables are defined as

$$2\tau - 2\gamma = u, \quad (19)$$

$$\tau = \frac{u+2\gamma}{2}, \quad (20)$$

and

$$d\tau = \frac{du}{2}. \quad (21)$$

The new form of the Eq. (18) is written as

$$I = \frac{1}{4\gamma\alpha}\int_{-\infty}^{\infty} \delta(u)e^{-\frac{\left(\frac{2\beta-u-2\gamma}{2}\right)^2}{\alpha}}du \quad (22)$$

by using Eq. (19), (20) and (21). Hence, the ultimate form of the Eq. (22) is found as

$$I = \frac{1}{4\gamma\alpha}e^{-\frac{(\beta-\gamma)^2}{\alpha}} \quad (23)$$

by inserting the $u - 0$ instead of u in Eq. (22). As can be seen from Eq. (23), the result is not the same as the Eq. (1) which is given (Gradshteyn and Ryzhik, 2007).

3. Conclusions

In this study, an integral solution used to obtain the Bessel-Gauss beam was revisited. Hankel transform and convolutional theorem were used to evaluate the integral. The reached solution of Bessel-Gauss beam expression from the analytical calculation differs from what is familiar in the literature. This alternative expression does not return the zero value when *z* is set to zero. Therefore, the derived form may be used to analyze the Bessel-Gauss beam for different optical propagation or scattering scenarios.

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